

THE PRIMITIVE IDEAL SPACE OF THE PARTIAL-ISOMETRIC CROSSED PRODUCT OF A SYSTEM BY A SINGLE AUTOMORPHISM

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ABSTRACT. Let (A, α) be a system consisting of a C^* -algebra A and an automorphism α of A . We describe the primitive ideal space of the partial-isometric crossed product $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ of the system by using its realization as a full corner of a classical crossed product and applying some results of Williams and Echterhoff.

1. INTRODUCTION

Lindiarni and Raeburn in [8] introduced the partial-isometric crossed product of a dynamical system (A, Γ^+, α) in which Γ^+ is the positive cone of a totally ordered abelian group Γ and α is an action of Γ^+ by endomorphisms of A . Note that since the C^* -algebra A is not necessarily unital, we require that each endomorphism α_s extends to a strictly continuous endomorphism $\overline{\alpha}_s$ of the multiplier algebra $M(A)$. This for an endomorphism α of A happens if and only if there exists an approximate identity (a_λ) in A and a projection $p \in M(A)$ such that $\alpha(a_\lambda)$ converges strictly to p in $M(A)$. We stress that if α is extendible, then we may not have $\overline{\alpha}(1_{M(A)}) = 1_{M(A)}$. A covariant representation of the system (A, Γ^+, α) is defined for which the endomorphisms α_s are implemented by partial isometries, and the associated partial-isometric crossed product $A \times_{\alpha}^{\text{piso}} \Gamma^+$ of the system is a C^* -algebra generated by a universal covariant representation such that there is a bijection between covariant representations of the system and nondegenerate representations of $A \times_{\alpha}^{\text{piso}} \Gamma^+$. This generalizes the covariant isometric representation theory that uses isometries to represent the semigroup of endomorphisms in a covariant representation of the system (see [3]). The authors of [8], in particular, studied the structure of the partial-isometric crossed product of the distinguished system $(B_{\Gamma^+}, \Gamma^+, \tau)$, where the action τ of Γ^+ on the subalgebra B_{Γ^+} of $\ell^\infty(\Gamma^+)$ is given by the right translation. Later, in [4], the authors showed that $A \times_{\alpha}^{\text{piso}} \Gamma^+$ is a full corner in a subalgebra of the C^* -algebra $\mathcal{L}(\ell^2(\Gamma^+) \otimes A)$ of adjointable operators on the Hilbert A -module $\ell^2(\Gamma^+) \otimes A \simeq \ell^2(\Gamma^+, A)$. This realization led them to identify the kernel of the natural homomorphism $q : A \times_{\alpha}^{\text{piso}} \Gamma^+ \rightarrow A \times_{\alpha}^{\text{iso}} \Gamma^+$ as a full corner of the compact operators $\mathcal{K}(\ell^2(\mathbb{N}) \otimes A)$, when Γ^+ is $\mathbb{N} := \mathbb{Z}^+$. So as an application, they recovered the Pimsner-Voiculescu exact sequence in [10]. Then in their subsequent work [5], they proved that for an extendible α -invariant ideal I of A (see the definition in [1]), the partial-isometric crossed product $I \times_{\alpha}^{\text{piso}} \Gamma^+$ sits naturally as an ideal in $A \times_{\alpha}^{\text{piso}} \Gamma^+$ such that $(A \times_{\alpha}^{\text{piso}} \Gamma^+) / (I \times_{\alpha}^{\text{piso}} \Gamma^+) \simeq A/I \times_{\alpha}^{\text{piso}} \Gamma^+$. This is actually a generalization of [2, Theorem 2.2]. They then combined these

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results to show that the large commutative diagram of [8, Theorem 5.6] associated to the system $(B_{\Gamma^+}, \Gamma^+, \tau)$ is valid for any totally ordered abelian group, not only for subgroups of \mathbb{R} . In particular, they use this large commutative diagram for $\Gamma^+ = \mathbb{N}$ to describe the ideal structure of the algebra $B_{\mathbb{N}} \times_{\tau}^{\text{piso}} \mathbb{N}$ explicitly.

Now here we consider a system (A, α) consisting of a C^* -algebra A and an automorphism α of A . So we actually have an action of the positive cone $\mathbb{N} = \mathbb{Z}^+$ of integers \mathbb{Z} by automorphisms of A . In the present work, we want to study $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$, the primitive ideal space of the partial-isometric crossed product $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ of the system. Since $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is in fact a full corner of the classical crossed product $(B_{\mathbb{Z}} \otimes A) \times \mathbb{Z}$ (see [4, §5]), $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$ is homeomorphic to $\text{Prim}((B_{\mathbb{Z}} \otimes A) \times \mathbb{Z})$. Therefore it is enough to describe $\text{Prim}((B_{\mathbb{Z}} \otimes A) \times \mathbb{Z})$. To do this, we apply the results on describing the primitive ideal space (ideal structure) of the classical crossed products from [12, 6]. So we consider the following two conditions:

- (1) when A is separable and abelian;
- (2) when A is separable and \mathbb{Z} acts on $\text{Prim } A$ freely (see §2).

For the first condition, by applying a theorem of Williams, $\text{Prim}((B_{\mathbb{Z}} \otimes A) \times \mathbb{Z})$ is homeomorphic to a quotient space of $\Omega(B_{\mathbb{Z}}) \times \Omega(A) \times \mathbb{T}$, where $\Omega(B_{\mathbb{Z}})$ and $\Omega(A)$ are the spectrums of the C^* -algebras $B_{\mathbb{Z}}$ and A respectively (recall that the dual $\hat{\mathbb{Z}}$ is identified with \mathbb{T} via the map $z \mapsto (\gamma_z : n \mapsto z^n)$). By computing $\Omega(B_{\mathbb{Z}})$, we parameterize the quotient space as a disjoint union, and then we precisely identify the open sets. For the second condition, we apply a result of Echterhoff which shows that $\text{Prim}((B_{\mathbb{Z}} \otimes A) \times \mathbb{Z})$ is homeomorphic to the quasi-orbit space of $\text{Prim}(B_{\mathbb{Z}} \otimes A) = \text{Prim } B_{\mathbb{Z}} \times \text{Prim } A$ (see in §2 that this is a quotient space of $\text{Prim}(B_{\mathbb{Z}} \otimes A)$). Again by a similar argument to the first condition, we describe the quotient space and its topology precisely.

We begin with a preliminary section in which we recall the theory of the partial-isometric crossed products, and some discussions on the primitive ideal space of the classical crossed products briefly. In section 3, for a system (A, α) consisting of a C^* -algebra A and an automorphism α of A , we apply the works of Williams and Echterhoff to describe $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$ using the realization of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ as a full corner of the classical crossed product $(B_{\mathbb{Z}} \otimes A) \times \mathbb{Z}$. As some examples, we compute the primitive ideal space of $C(\mathbb{T}) \times_{\alpha}^{\text{piso}} \mathbb{N}$ where the action α is given by rotation through the angle $2\pi\theta$ with θ rational and irrational. Moreover the description of the primitive ideal space of the Pimsner-Voiculescu Toeplitz algebra associated to the system (A, α) is completely obtained, as it is isomorphic to $A \times_{\alpha^{-1}}^{\text{piso}} \mathbb{N}$. We also discuss necessary and sufficient conditions under which $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is GCR (postliminal or type I). Finally in the last section, we discuss the primitivity and simplicity of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$.

2. PRELIMINARIES

2.1. The partial-isometric crossed product. A *partial-isometric representation* of \mathbb{N} on a Hilbert space H is a map $V : \mathbb{N} \rightarrow B(H)$ such that each $V_n := V(n)$ is a partial isometry, and $V_{n+m} = V_n V_m$ for all $n, m \in \mathbb{N}$.

A *covariant partial-isometric representation* of (A, α) on a Hilbert space H is a pair (π, V) consisting of a nondegenerate representation $\pi : A \rightarrow B(H)$ and a partial-isometric representation $V : \mathbb{N} \rightarrow B(H)$ such that

$$(2.1) \quad \pi(\alpha_n(a)) = V_n \pi(a) V_n^* \quad \text{and} \quad V_n^* V_n \pi(a) = \pi(a) V_n^* V_n$$

for all $a \in A$ and $n \in \mathbb{N}$.

Note that every system (A, α) admits a nontrivial covariant partial-isometric representation [8, Example 4.6]: let π be a nondegenerate representation of A on H . Define $\Pi : A \rightarrow B(\ell^2(\mathbb{N}, H))$ by $(\Pi(a)\xi)(n) = \pi(\alpha_n(a))\xi(n)$. If

$$\mathcal{H} := \overline{\text{span}}\{\xi \in \ell^2(\mathbb{N}, H) : \xi(n) \in \overline{\pi(\alpha_n(1))}H \text{ for all } n\},$$

then the representation Π is nondegenerate on \mathcal{H} . Now for every $m \in \mathbb{N}$, define V_m on \mathcal{H} by $(V_m\xi)(n) = \xi(n+m)$. Then the pair $(\Pi|_{\mathcal{H}}, V)$ is a partial-isometric covariant representation of (A, α) on \mathcal{H} . One can see that if we take π faithful, then Π will be faithful as well, and $\mathcal{H} = \ell^2(\mathbb{N}, H)$ whenever $\overline{\alpha}(1) = 1$ (e.g. when α is an automorphism).

Definition 2.1. A partial-isometric crossed product of (A, α) is a triple $(B, j_A, j_{\mathbb{N}})$ consisting of a C^* -algebra B , a nondegenerate homomorphism $i_A : A \rightarrow B$, and a partial-isometric representation $i_{\mathbb{N}} : \mathbb{N} \rightarrow M(B)$ such that:

- (i) the pair $(j_A, j_{\mathbb{N}})$ is a covariant representation of (A, α) in B ;
- (ii) for every covariant partial-isometric representation (π, V) of (A, α) on a Hilbert space H , there exists a nondegenerate representation $\pi \times V : B \rightarrow B(H)$ such that $(\pi \times V) \circ i_A = \pi$ and $(\overline{\pi \times V}) \circ i_{\mathbb{N}} = V$; and
- (iii) the C^* -algebra B is spanned by $\{i_{\mathbb{N}}(n)^* i_A(a) i_{\mathbb{N}}(m) : n, m \in \mathbb{N}, a \in A\}$.

By [8, Proposition 4.7], the partial-isometric crossed product of (A, α) always exists, and it is unique up to isomorphism. Thus we write the partial-isometric crossed product B as $A \times_{\alpha}^{\text{piso}} \mathbb{N}$.

We recall that by [8, Theorem 4.8], a covariant representation (π, V) of (A, α) on H induces a faithful representation $\pi \times V$ of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ if and only if π is faithful on the range of $(1 - V_n^* V_n)$ for every $n > 0$ (one can actually see that it is enough to verify that π is faithful on the range of $(1 - V^* V)$, where $V := V_1$).

2.2. The primitive ideal space of crossed products associated to second countable locally compact transformation groups.

Let Γ be a discrete group which acts on a topological space X . For every $x \in X$, the set $\Gamma \cdot x := \{s \cdot x : s \in \Gamma\}$ is called the Γ -orbit of x . The set $\Gamma_x := \{s \in \Gamma : s \cdot x = x\}$, which is a subgroup of Γ , is called the *stability group* of x . We say the Γ -action is *free* or Γ acts on X *freely* if $\Gamma_x = \{e\}$ for all $x \in X$. Consider a relation \sim on X such that for $x, y \in X$, $x \sim y$ if and only if $\overline{\Gamma \cdot x} = \overline{\Gamma \cdot y}$. One can see that this is an equivalence relation on X . The set of all equivalence classes equipped with the quotient topology is denoted by $\mathcal{O}(X)$ and called the *quasi-orbit space*, which is always a T_0 -topological space. The equivalence class of each $x \in X$ is denoted by $\mathcal{O}(x)$ and called the *quasi-orbit* of x .

Now let Γ be an abelian countable discrete group which acts on a second countable locally compact Hausdorff space X . So (Γ, X) is a second countable locally compact transformation group with Γ abelian. Then the associated dynamical system $(C_0(X), \Gamma, \tau)$ is separable with Γ abelian, and so the primitive ideals of $C_0(X) \rtimes_{\tau} \Gamma$ are known (see [12, Theorem 8.21]). Furthermore, the topology of $\text{Prim}(C_0(X) \rtimes_{\tau} \Gamma)$ has been beautifully described [12, Theorem 8.39]. So here we want to recall the discussion on $\text{Prim}(C_0(X) \rtimes_{\tau} \Gamma)$ in brief. See more in [12] that this is indeed a huge and deep discussion.

Let N be a subgroup of Γ . If we restrict the action τ to N , then we obtain a dynamical system $(C_0(X), N, \tau|_N)$ with the associated crossed product $C_0(X) \rtimes_{\tau|_N} N$. Suppose that \mathbf{X}_N^Γ is the Green's $((C_0(X) \otimes C_0(\Gamma/N)) \rtimes_{\tau \otimes \text{lt}} \Gamma) - (C_0(X) \rtimes_{\tau|_N} N)$ -imprimitivity bimodule whose structure can be found in [12, Theorem 4.22]. If (π, V) is a covariant representation of $(C_0(X), N, \tau|_N)$, then $\text{Ind}_N^\Gamma(\pi \times V)$ denotes the representation of $C_0(X) \rtimes_\tau \Gamma$ induced from the representation $\pi \times V$ of $C_0(X) \rtimes_{\tau|_N} N$ via \mathbf{X}_N^Γ . Now for $x \in X$, let $\varepsilon_x : C_0(X) \rightarrow \mathbb{C} \simeq B(\mathbb{C})$ be the evaluation map at x and w a character of Γ_x . Then the pair (ε_x, w) is a covariant representation of $(C_0(X), \Gamma_x, \tau|_{\Gamma_x})$ such that the associated representation $\varepsilon_x \times w$ of $C_0(X) \rtimes \Gamma_x$ is irreducible, and hence by [12, Proposition 8.27], $\text{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times w)$ is an irreducible representation of $C_0(X) \rtimes_\tau \Gamma$. So $\ker(\text{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times w))$ is a primitive ideal of $C_0(X) \rtimes_\tau \Gamma$. Note if a primitive ideal is obtained in this way, then we say it is *induced from a stability group*. In fact by [12, Theorem 8.21], all primitive ideals of $C_0(X) \rtimes_\tau \Gamma$ are induced from stability groups. Moreover since for every $w \in \widehat{\Gamma_x}$ there is a $\gamma \in \widehat{\Gamma}$ such that $w = \gamma|_{\Gamma_x}$, every primitive ideal of $C_0(X) \rtimes_\tau \Gamma$ is actually given by the kernel of an induced irreducible representation $\text{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times \gamma|_{\Gamma_x})$ correspondent to a pair (x, γ) in $X \times \widehat{\Gamma}$. To see the description of the topology of $\text{Prim}(C_0(X) \rtimes_\tau \Gamma)$, first note that if (x, γ) and (y, μ) belong to $X \times \widehat{\Gamma}$ such that $\overline{\Gamma \cdot x} = \overline{\Gamma \cdot y}$ (which implies that $\Gamma_x = \Gamma_y$) and $\gamma|_{\Gamma_x} = \mu|_{\Gamma_x}$, then by [12, Lemma 8.34],

$$\ker(\text{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times \gamma|_{\Gamma_x})) = \ker(\text{Ind}_{\Gamma_y}^\Gamma(\varepsilon_y \times \mu|_{\Gamma_y})).$$

So define a relation on $X \times \widehat{\Gamma}$ such that $(x, \gamma) \sim (y, \mu)$ if

$$(2.2) \quad \overline{\Gamma \cdot x} = \overline{\Gamma \cdot y} \quad \text{and} \quad \gamma|_{\Gamma_x} = \mu|_{\Gamma_x}.$$

One can see that \sim is an equivalence relation on $X \times \widehat{\Gamma}$. Now consider the quotient space $X \times \widehat{\Gamma} / \sim$ equipped with the quotient topology. Then we have:

Theorem 2.2. [12, Theorem 8.39] *Let (Γ, X) be a second countable locally compact transformation group with Γ abelian. Then the map $\Phi : X \times \widehat{\Gamma} \rightarrow \text{Prim}(C_0(X) \rtimes_\tau \Gamma)$ defined by*

$$\Phi(x, \gamma) := \ker(\text{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times \gamma|_{\Gamma_x}))$$

is a continuous and open surjection, and factors through a homeomorphism of $X \times \widehat{\Gamma} / \sim$ onto $\text{Prim}(C_0(X) \rtimes_\tau \Gamma)$.

Remark 2.3. In the theorem above, note that $\text{Prim}(C_0(X) \rtimes_\tau \Gamma)$ is then a second countable space. This is because as it is mentioned in [12, Remark 8.40], the quotient map $\mathbf{q} : X \times \widehat{\Gamma} \rightarrow X \times \widehat{\Gamma} / \sim$ is open. Moreover, X and $\widehat{\Gamma}$ both are second countable.

Theorem 2.2 can be applied to see that the primitive ideal space of the rational rotation algebra is homeomorphic to \mathbb{T}^2 . We skip it here and refer readers to [12, Example 8.45] for more details.

2.3. The primitive ideal space of crossed products by free actions. Let (A, Γ, α) be a classical dynamical system with Γ discrete. Then the system gives an action of Γ on the spectrum \hat{A} of A by $s \cdot [\pi] := [\pi \circ \alpha_s^{-1}]$ for every $s \in \Gamma$ and $[\pi] \in \hat{A}$ (see [12, Lemma 2.8] and [11, Lemma 7.1]). This also induces an action of Γ on $\text{Prim } A$ such that $s \cdot P := \alpha_s(P)$ for each $s \in \Gamma$ and $P \in \text{Prim } A$.

Recall that if π is a (nondegenerate) representation of A on H with $\ker \pi = J$, then $\text{Ind } \pi$ denotes the induced representation $\tilde{\pi} \times U$ of $A \times_\alpha \Gamma$ on $\ell^2(\Gamma, H)$ associated to the covariant pair $(\tilde{\pi}, U)$ of (A, Γ, α) defined by

$$(\tilde{\pi}(a)\xi)(s) = \pi(\alpha_s^{-1}(a))\xi(s) \quad \text{and} \quad (U_t\xi)(s) = \xi(t^{-1}s)$$

for all every $a \in A$, $\xi \in \ell^2(\Gamma, H)$, and $s, t \in \Gamma$. Note that by $\text{Ind } J$, we mean $\ker(\text{Ind } \pi)$.

Now let (A, Γ, α) be a classical dynamical system in which A is separable and Γ is an abelian discrete countable group. If Γ acts on $\text{Prim } A$ freely, then each primitive ideal $\ker \pi = P$ of A induces a primitive ideal of $A \times_\alpha \Gamma$, namely $\text{Ind } P = \ker(\text{Ind } \pi)$, and the description of $\text{Prim}(A \times_\alpha \Gamma)$ is completely available:

Theorem 2.4. [6, Corollary 10.16] *Suppose in the system (A, Γ, α) that A is separable and Γ is an amenable discrete countable group. If Γ acts on $\text{Prim } A$ freely, then the map*

$$\begin{aligned} \mathcal{O}(\text{Prim } A) &\rightarrow \text{Prim}(A \times_\alpha \Gamma) \\ \mathcal{O}(P) &\mapsto \text{Ind } P = \ker(\text{Ind } \pi) \end{aligned}$$

is a homeomorphism, where π is an irreducible representation of A with $\ker \pi = P$. In particular, $A \times_\alpha \Gamma$ is simple if and only if every Γ -orbit is dense in $\text{Prim } A$.

We can apply the above Theorem to see that the irrational rotation algebras are simple. Readers can refer to [6, Example 10.18] or [12, Example 8.46] for more details.

3. THE PRIMITIVE IDEAL SPACE OF $A \times_\alpha^{\text{piso}} \mathbb{N}$ BY AUTOMORPHIC ACTION

First recall that if T is the isometry in $B(\ell^2(\mathbb{N}))$ such that $T(e_n) = e_{n+1}$ on the usual orthonormal basis $\{e_n\}_{n=0}^\infty$ of $\ell^2(\mathbb{N})$, then we have

$$\mathcal{K}(\ell^2(\mathbb{N})) = \overline{\text{span}}\{T_n(1 - TT^*)T_m^* : n, m \in \mathbb{N}\}.$$

Now consider a system (A, α) consisting of a C^* -algebra A and an automorphism α of A . Let the triples $(A \times_\alpha^{\text{piso}} \mathbb{N}, j_A, v)$ and $(A \times_\alpha \mathbb{Z}, i_A, u)$ be the partial-isometric crossed product and the classical crossed product of the system respectively. Here our goal is to describe the primitive ideal space of $A \times_\alpha^{\text{piso}} \mathbb{N}$ and its topology completely. See in [4] that the kernel of the natural homomorphism $q : (A \times_\alpha^{\text{piso}} \mathbb{N}, j_A, v) \rightarrow (A \times_\alpha \mathbb{Z}, i_A, u)$ given by $q(v_n^* j_A(a) v_m) = u_n^* i_A(a) u_m$, is isomorphic to the algebra of compact operators $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$. Therefore we have a short exact sequence

$$(3.1) \quad 0 \longrightarrow (\mathcal{K}(\ell^2(\mathbb{N})) \otimes A) \xrightarrow{\mu} A \times_\alpha^{\text{piso}} \mathbb{N} \xrightarrow{q} A \times_\alpha \mathbb{Z} \longrightarrow 0,$$

where $\mu(T_n(1 - TT^*)T_m^* \otimes a) = v_n^* j_A(a)(1 - v^* v)v_m$ for all $a \in A$ and $n, m \in \mathbb{N}$. So $\text{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N})$ as a set, is given by the sets $\text{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$ and $\text{Prim}(A \times_\alpha \mathbb{Z})$. With no condition on the system, we do not have much information about $\text{Prim}(A \times_\alpha \mathbb{Z})$ in general. However, by [4, Proposition 2.5], we do know that $\ker q \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ is an essential ideal of $A \times_\alpha^{\text{piso}} \mathbb{N}$. Therefore $\text{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$ which is homeomorphic to $\text{Prim } A$, sits in $\text{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N})$ as an open dense subset. We will identify this open dense subset, namely the primitive ideals $\{\mathcal{I}_P : P \in \text{Prim } A\}$ of $\text{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N})$ coming from $\text{Prim } A$, shortly. Moreover see in [4, §5] that $A \times_\alpha^{\text{piso}} \mathbb{N}$ is a full corner of the classical crossed product $(B_{\mathbb{Z}} \otimes A) \rtimes_{\beta \otimes \alpha^{-1}} \mathbb{Z}$, where $B_{\mathbb{Z}} := \overline{\text{span}}\{1_n : n \in \mathbb{Z}\} \subset \ell^\infty(\mathbb{Z})$, and the action β of \mathbb{Z} on $B_{\mathbb{Z}}$ is given by translation such that $\beta_m(1_n) = 1_{n+m}$ for all $m, n \in \mathbb{Z}$. Thus $\text{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N})$ is homeomorphic to $\text{Prim}((B_{\mathbb{Z}} \otimes A) \rtimes_{\beta \otimes \alpha^{-1}} \mathbb{Z})$, and

hence it suffices to describe $\text{Prim}((B_{\mathbb{Z}} \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z})$ and its topology. To do this, we will consider two conditions on the system that make us able to apply a theorem of Williams and a result by Echterhoff. We will also identify those primitive ideals of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ coming from $\text{Prim}(A \times_{\alpha} \mathbb{Z})$, which form a closed subset of $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$. But first, let us identify the primitive ideals \mathcal{I}_P .

Proposition 3.1. *Let $\pi : A \rightarrow B(H)$ be a nonzero irreducible representation of A with $P := \ker \pi$. If the pair (Π, V) is defined as in [8, Example 4.6] (see §2), then the associated representation of $(A \times_{\alpha}^{\text{piso}} \mathbb{N}, j_A, v)$, which we denote by $(\Pi \times V)_P$, is irreducible on $\ell^2(\mathbb{N}, H)$, and does not vanish on $\ker q \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes A$.*

Proof. To see that $(\Pi \times V)_P$ is irreducible, we show that every $\xi \in \ell^2(\mathbb{N}, H) \setminus \{0\}$ is a cyclic vector for $(\Pi \times V)_P$, that is $\ell^2(\mathbb{N}, H) = \overline{\text{span}}\{(\Pi \times V)_P(x)(\xi) : x \in (A \times_{\alpha}^{\text{piso}} \mathbb{N})\}$. We show that

$$(3.2) \quad \mathcal{H} := \overline{\text{span}}\{(\Pi \times V)_P(v_n^* j_A(a)(1 - v^* v)v_m)(\xi) : a \in A, n, m \in \mathbb{N}\}$$

equals $\ell^2(\mathbb{N}, H)$ which is enough. By viewing $\ell^2(\mathbb{N}, H)$ as the Hilbert space $\ell^2(\mathbb{N}) \otimes H$, it suffices to see that each $e_n \otimes h$ belongs to \mathcal{H} , where $\{e_n\}_{n=0}^{\infty}$ is the usual orthonormal basis of $\ell^2(\mathbb{N})$ and $h \in H$. Since $\xi \neq 0$ in $\ell^2(\mathbb{N}, H)$, there is $m \in \mathbb{N}$ such that $\xi(m) \neq 0$ in H . But $\xi(m)$ is a cyclic vector for the representation $\pi : A \rightarrow B(H)$ as π is irreducible. Thus we have $\overline{\text{span}}\{\pi(a)(\xi(m)) : a \in A\} = H$, and hence $\text{span}\{e_n \otimes (\pi(a)\xi(m)) : n \in \mathbb{N}, a \in A\}$ is dense in $\ell^2(\mathbb{N}) \otimes H \simeq \ell^2(\mathbb{N}, H)$. So we only have to show that \mathcal{H} contains each element $e_n \otimes (\pi(a)\xi(m))$. Calculation shows that

$$\begin{aligned} e_n \otimes (\pi(a)\xi(m)) &= (V_n^* \Pi(a)(1 - V^* V)V_m)(\xi) \\ &= (\Pi \times V)_P(v_n^* j_A(a)(1 - v^* v)v_m)(\xi), \end{aligned}$$

and therefore $e_n \otimes (\pi(a)\xi(m)) \in \mathcal{H}$ for every $a \in A$ and $n \in \mathbb{N}$. So we have $\mathcal{H} = \ell^2(\mathbb{N}, H)$.

To show that $(\Pi \times V)_P$ does not vanish on $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$, first note that since π is nonzero, $\pi(a)h \neq 0$ for some $a \in A$ and $h \in H$. Now if we take $(1 - TT^*) \otimes a \in \mathcal{K}(\ell^2(\mathbb{N})) \otimes A$, then $(\Pi \times V)_P(\mu((1 - TT^*) \otimes a)) = (\Pi \times V)_P(j(a)(1 - v^* v)) \neq 0$. This is because for $(e_0 \otimes h) \in \ell^2(\mathbb{N}, H)$, we have

$$(\Pi \times V)_P(j_A(a)(1 - v^* v))(e_0 \otimes h) = \Pi(a)(1 - V^* V)(e_0 \otimes h) = e_0 \otimes \pi(a)h,$$

which is not zero in $\ell^2(\mathbb{N}, H)$ as $\pi(a)h \neq 0$. \square

Remark 3.2. The primitive ideals \mathcal{I}_P are actually the kernels of the irreducible representations $(\Pi \times V)_P$ which form the open dense subset

$$\mathcal{U} := \{\mathcal{I} \in \text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}) : \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q \not\subset \mathcal{I}\}$$

of $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$ homeomorphic to $\text{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$. Now $\text{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$ itself is homeomorphic to $\text{Prim} A$ via the (Rieffel) homeomorphism $P \mapsto \mathcal{K}(\ell^2(\mathbb{N})) \otimes P$. But $\mathcal{K}(\ell^2(\mathbb{N})) \otimes P$ is the kernel of the irreducible representation $(\text{id} \otimes \pi)$ of $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$, which indeed equals the restriction $(\Pi \times V)_P|_{\mathcal{K}(\ell^2(\mathbb{N})) \otimes A}$. Therefore we have

$$\begin{aligned} \mathcal{I}_P \cap (\mathcal{K}(\ell^2(\mathbb{N})) \otimes A) &= \ker((\Pi \times V)_P|_{\mathcal{K}(\ell^2(\mathbb{N})) \otimes A}) \\ &= \ker(\text{id} \otimes \pi) = \mathcal{K}(\ell^2(\mathbb{N})) \otimes P. \end{aligned}$$

Consequently the map $P \mapsto \mathcal{I}_P$ is a homeomorphism of $\text{Prim} A$ onto the open dense subset \mathcal{U} of $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$.

Now we want to describe the topology of $\text{Prim}((B_{\mathbb{Z}} \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z}) \simeq \text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$ and identify the primitive ideals of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ coming from $A \times_{\alpha} \mathbb{Z}$ under the following two conditions:

- (1) when A is separable and abelian, by applying a theorem of Williams, namely Theorem 2.2;
- (2) when A is separable and \mathbb{Z} acts on $\text{Prim } A$ freely, by applying Theorem 2.4.

3.1. The topology of $\text{Prim}((B_{\mathbb{Z}} \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z})$ when A is separable and abelian. Suppose that A is separable and abelian. Then $(B_{\mathbb{Z}} \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z}$ is isomorphic to the crossed product $C_0(\Omega(B_{\mathbb{Z}} \otimes A)) \times_{\tau} \mathbb{Z}$ associated to the second countable locally compact transformation group $(\mathbb{Z}, \Omega(B_{\mathbb{Z}} \otimes A))$. Therefore by Theorem 2.2, $\text{Prim}((B_{\mathbb{Z}} \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z})$ is homeomorphic to $\Omega(B_{\mathbb{Z}} \otimes A) \times \mathbb{T} / \sim$. But we want to describe $\Omega(B_{\mathbb{Z}} \otimes A) \times \mathbb{T} / \sim$ precisely. To do this, we need to analyze $\Omega(B_{\mathbb{Z}} \otimes A)$, and since $\Omega(B_{\mathbb{Z}} \otimes A) \simeq \Omega(B_{\mathbb{Z}}) \times \Omega(A)$ (see [11, Theorem B.37] or [11, Theorem B.45]), we have to compute $\Omega(B_{\mathbb{Z}})$ first.

Lemma 3.3. *Let $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ be the two-point compactification of \mathbb{Z} . Then $\Omega(B_{\mathbb{Z}})$ is homeomorphic to the open dense subset $\mathbb{Z} \cup \{\infty\}$.*

Proof. First note that $B_{\mathbb{Z}}$ exactly consists of those functions $f : \mathbb{Z} \rightarrow \mathbb{C}$ such that $\lim_{n \rightarrow -\infty} f(n) = 0$ and $\lim_{n \rightarrow \infty} f(n)$ exists. Thus the complex homomorphisms (irreducible representations) of $B_{\mathbb{Z}}$ are given by the evaluation maps $\{\varepsilon_n : n \in \mathbb{Z}\}$, and the map $\varepsilon_{\infty} : B_{\mathbb{Z}} \rightarrow \mathbb{C}$ defined by $\varepsilon_{\infty}(f) := \lim_{n \rightarrow \infty} f(n)$ for all $f \in B_{\mathbb{Z}}$. So we have $\Omega(B_{\mathbb{Z}}) = \{\varepsilon_n : n \in \mathbb{Z}\} \cup \{\varepsilon_{\infty}\}$. Note that the kernel of ε_{∞} is the ideal $C_0(\mathbb{Z}) = \overline{\text{span}}\{1_n - 1_m : n < m \in \mathbb{Z}\}$ of $B_{\mathbb{Z}}$. Now let $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ be the two-point compactification of \mathbb{Z} which is homeomorphic to the subspace

$$X := \{-1\} \cup \{-1 + 1/(1 - n) : n \in \mathbb{Z}, n < 0\} \cup \{1 - 1/(1 + n) : n \in \mathbb{Z}, n \geq 0\} \cup \{1\}$$

of \mathbb{R} . Then the map

$$f \in B_{\mathbb{Z}} \mapsto \tilde{f} \in C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}),$$

where

$$\tilde{f}(r) := \begin{cases} \lim_{n \rightarrow \infty} f(n) & \text{if } r = \infty, \\ f(r) & \text{if } r \in \mathbb{Z}, \text{ and} \\ 0 & \text{if } r = -\infty, \end{cases}$$

embeds $B_{\mathbb{Z}}$ in $C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})$ as the maximal ideal

$$I := \{\tilde{f} \in C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}) : \tilde{f}(-\infty) = 0\}.$$

Thus it follows that $\Omega(B_{\mathbb{Z}})$ is homeomorphic to \hat{I} , and \hat{I} itself is homeomorphic to the open subset

$$\{\pi \in C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})^{\wedge} : \pi|_I \neq 0\} = \{\tilde{\varepsilon}_r : r \in (\mathbb{Z} \cup \{\infty\})\}$$

of $C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})^{\wedge}$ in which each $\tilde{\varepsilon}_r$ is an evaluation map. So by the homeomorphism between $C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})^{\wedge}$ and $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, the open subset $\{\tilde{\varepsilon}_r : r \in (\mathbb{Z} \cup \{\infty\})\}$ is homeomorphic to the open (dense) subset $\mathbb{Z} \cup \{\infty\}$ of $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ equipped with the relative topology. Therefore $\Omega(B_{\mathbb{Z}})$ is in fact

homeomorphic to $\mathbb{Z} \cup \{\infty\}$. One can see that $\mathbb{Z} \cup \{\infty\}$ is indeed a second countable locally compact Hausdorff space with

$$\mathcal{B} := \{\{n\} : n \in \mathbb{Z}\} \cup \{J_n : n \in \mathbb{Z}\}$$

as a countable basis for its topology, where $J_n := \{n, n+1, n+2, \dots\} \cup \{\infty\}$ for every $n \in \mathbb{Z}$. \square

Remark 3.4. Before we continue, we need to mention that, if A is a separable C^* -algebra (not necessarily abelian), then by [11, Theorem B.45] and using Lemma 3.3, $(C_0(\mathbb{Z}) \otimes A)^\wedge$ and $(B_{\mathbb{Z}} \otimes A)^\wedge$ are homeomorphic to $\mathbb{Z} \times \hat{A}$ and $(\mathbb{Z} \cup \{\infty\}) \times \hat{A}$ respectively. Also $\text{Prim}(C_0(\mathbb{Z}) \otimes A)$ and $\text{Prim}(B_{\mathbb{Z}} \otimes A)$ are homeomorphic to $\mathbb{Z} \times \text{Prim } A$ and $(\mathbb{Z} \cup \{\infty\}) \times \text{Prim } A$ respectively (note that these homeomorphisms are \mathbb{Z} -equivariant for the action of \mathbb{Z}). Since $C_0(\mathbb{Z}) \otimes A$ is an (essential) ideal of $B_{\mathbb{Z}} \otimes A$, we have the following commutative diagram:

$$\begin{array}{ccccccc} \mathbb{Z} \times \hat{A} & \longrightarrow & (C_0(\mathbb{Z}) \otimes A)^\wedge & \xrightarrow{\Theta} & \text{Prim}(C_0(\mathbb{Z}) \otimes A) & \longrightarrow & \mathbb{Z} \times \text{Prim } A \\ \text{id} \downarrow & & \downarrow \iota & & \downarrow \tilde{\iota} & & \downarrow \text{id} \\ (\mathbb{Z} \cup \{\infty\}) \times \hat{A} & \longrightarrow & (B_{\mathbb{Z}} \otimes A)^\wedge & \xrightarrow{\tilde{\Theta}} & \text{Prim}(B_{\mathbb{Z}} \otimes A) & \longrightarrow & (\mathbb{Z} \cup \{\infty\}) \times \text{Prim } A, \end{array}$$

where Θ and $\tilde{\Theta}$ are the canonical continuous, open surjections, and ι and $\tilde{\iota}$ are the canonical embedding maps. Now to see how \mathbb{Z} acts on $(\mathbb{Z} \cup \{\infty\}) \times \hat{A}$ (and accordingly on $(\mathbb{Z} \cup \{\infty\}) \times \text{Prim } A$), note that since the crossed products $(C_0(\mathbb{Z}) \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z}$ and $(C_0(\mathbb{Z}) \otimes A) \times_{\beta \otimes \text{id}} \mathbb{Z}$ are isomorphic (see [12, Lemma 7.4]), we have

$$n \cdot (m, [\pi]) = (m+n, [\pi]) \text{ and } n \cdot (\infty, [\pi]) = (n+\infty, n \cdot [\pi]) = (\infty, [\pi \circ \alpha_n])$$

for all $n, m \in \mathbb{Z}$ and $[\pi] \in \hat{A}$. Accordingly

$$n \cdot (m, P) = (m+n, P) \text{ and } n \cdot (\infty, P) = (\infty, \alpha_n^{-1}(P))$$

for all $n, m \in \mathbb{Z}$ and $P \in \text{Prim } A$.

So when A is separable and abelian, using Lemma 3.3, $\Omega(B_{\mathbb{Z}} \otimes A) = (\mathbb{Z} \cup \{\infty\}) \times \Omega(A)$. Now to describe $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} / \sim$, note that by Remark 3.4, \mathbb{Z} acts on $(\mathbb{Z} \cup \{\infty\}) \times \Omega(A)$ as follows:

$$n \cdot (m, \phi) = (m+n, \phi) \text{ and } n \cdot (\infty, \phi) = (\infty, \phi \circ \alpha_n)$$

for all $n, m \in \mathbb{Z}$ and $\phi \in \Omega(A)$. Therefore, the stability group of each (m, ϕ) is $\{0\}$, and the stability group of each (∞, ϕ) equals the stability group \mathbb{Z}_ϕ of ϕ . Accordingly, the \mathbb{Z} -orbit of each (m, ϕ) is $\mathbb{Z} \times \{\phi\}$, and the \mathbb{Z} -orbit of (∞, ϕ) is $\{\infty\} \times \mathbb{Z} \cdot \phi$, where $\mathbb{Z} \cdot \phi$ is the \mathbb{Z} -orbit of ϕ . So for the pairs (or triples) $((m, \phi), z)$ and $((n, \psi), w)$ of $(\mathbb{Z} \times \Omega(A)) \times \mathbb{T}$, we have

$$\begin{aligned} ((m, \phi), z) \sim ((n, \psi), w) &\iff \overline{\mathbb{Z} \cdot (m, \phi)} = \overline{\mathbb{Z} \cdot (n, \psi)} \\ &\iff \overline{\mathbb{Z} \times \{\phi\}} = \overline{\mathbb{Z} \times \{\psi\}} \\ &\iff \overline{\mathbb{Z}} \times \overline{\{\phi\}} = \overline{\mathbb{Z}} \times \overline{\{\psi\}} \\ &\iff (\mathbb{Z} \cup \{\infty\}) \times \overline{\{\phi\}} = (\mathbb{Z} \cup \{\infty\}) \times \overline{\{\psi\}} \\ &\iff (\mathbb{Z} \cup \{\infty\}) \times \{\phi\} = (\mathbb{Z} \cup \{\infty\}) \times \{\psi\}. \end{aligned}$$

The last equivalence follows from the fact that $\Omega(A)$ is Hausdorff. Therefore $((m, \phi), z)$ and $((n, \psi), w)$ are in the same equivalence class in $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} / \sim$ if and only if $\phi = \psi$, while $((m, \phi), z) \sim ((\infty, \psi), w)$ for every $\psi \in \Omega(A)$ and $w \in \mathbb{T}$, because

$$\overline{\mathbb{Z} \cdot (\infty, \psi)} = \overline{\{\infty\} \times \mathbb{Z} \cdot \psi} = \overline{\{\infty\}} \times \overline{\mathbb{Z} \cdot \psi} = \{\infty\} \times \overline{\mathbb{Z} \cdot \psi}.$$

Thus if $\phi \in \Omega(A)$, then all pairs $((m, \phi), z)$ for every $m \in \mathbb{Z}$ and $z \in \mathbb{T}$ are in the same equivalence class, which can be parameterized by $\phi \in \Omega(A)$. On the other hand, for the pairs $((\infty, \phi), z)$ and $((\infty, \psi), w)$, we have

$$\begin{aligned} ((\infty, \phi), z) \sim ((\infty, \psi), w) &\iff \overline{\mathbb{Z} \cdot (\infty, \phi)} = \overline{\mathbb{Z} \cdot (\infty, \psi)} \text{ and } \gamma_z|_{\mathbb{Z}_\phi} = \gamma_w|_{\mathbb{Z}_\psi} \\ &\iff \{\infty\} \times \overline{\mathbb{Z} \cdot \phi} = \{\infty\} \times \overline{\mathbb{Z} \cdot \psi} \text{ and } \gamma_z|_{\mathbb{Z}_\phi} = \gamma_w|_{\mathbb{Z}_\psi}. \end{aligned}$$

Therefore

$$((\infty, \phi), z) \sim ((\infty, \psi), w) \iff \overline{\mathbb{Z} \cdot \phi} = \overline{\mathbb{Z} \cdot \psi} \text{ and } \gamma_z|_{\mathbb{Z}_\phi} = \gamma_w|_{\mathbb{Z}_\psi},$$

which means if and only if the pairs (ϕ, z) and (ψ, w) are in the same equivalence class in the quotient space $\Omega(A) \times \mathbb{T} / \sim$ homeomorphic to $\text{Prim}(A \times_\alpha \mathbb{Z})$. Therefore $((\infty, \phi), z) \sim ((\infty, \psi), w)$ in $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} / \sim$ precisely when $(\phi, z) \sim (\psi, w)$ in $\Omega(A) \times \mathbb{T} / \sim$, and hence the class of each $((\infty, \phi), z)$ in $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} / \sim$ can be parameterized by the class of (ϕ, z) in $\Omega(A) \times \mathbb{T} / \sim$. So we can identify $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} / \sim$ with the disjoint union

$$\Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim).$$

Now we have:

Theorem 3.5. *Let (A, α) be a system consisting of a separable abelian C^* -algebra A and an automorphism α of A . Then $\text{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N})$ is homeomorphic to $\Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim)$, equipped with the (quotient) topology in which the open sets are of the form*

$$\{U \subset \Omega(A) : U \text{ is open in } \Omega(A)\} \cup$$

$$\{U \cup W : U \text{ is a nonempty open subset of } \Omega(A), \text{ and } W \text{ is open in } (\Omega(A) \times \mathbb{T} / \sim)\}.$$

Proof. Since the quotient map $\mathbf{q} : ((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} \rightarrow \Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim)$ is open, as well as $\tilde{\mathbf{q}} : \Omega(A) \times \mathbb{T} \rightarrow \Omega(A) \times \mathbb{T} / \sim$, for every $n \in \mathbb{Z}$, every open subset O of $\Omega(A)$, and every open subset V of \mathbb{T} , the forward image of open subsets $\{n\} \times O \times V$ and $J_n \times O \times V$ by \mathbf{q} , forms a basis for the topology of $\Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim)$, which is

$$\{O \subset \Omega(A) : O \text{ is open in } \Omega(A)\} \cup$$

$$\{O \cup \tilde{\mathbf{q}}(O \times V) : O \text{ is a nonempty open subset of } \Omega(A), \text{ and } V \text{ is open in } \mathbb{T}\}.$$

As the open subsets $\tilde{\mathbf{q}}(O \times V)$ also form a basis for the quotient topology of $\Omega(A) \times \mathbb{T} / \sim$, we can see that each open subset of $\Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim)$ is either an open subset U of $\Omega(A)$ or of the form $U \cup W$ for some nonempty open subset U in $\Omega(A)$ and some open subset W in $\Omega(A) \times \mathbb{T} / \sim$. \square

Remark 3.6. Under the condition of Theorem 3.5, the primitive ideals of $\text{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N})$ coming from $\text{Prim}(A \times_\alpha \mathbb{Z})$, which form the closed subset

$$\mathcal{F} := \{\mathcal{J} \in \text{Prim}(A \times_\alpha^{\text{piso}} \mathbb{N}) : \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q \subset \mathcal{J}\},$$

are the kernels of the irreducible representations $(\text{Ind}_{\mathbb{Z}_\phi}^{\mathbb{Z}}(\phi \times \gamma_z|_{\mathbb{Z}_\phi})) \circ q$ corresponding to the equivalence classes of the pairs (ϕ, z) in $\Omega(A) \times \mathbb{T} / \sim$ (again by using Theorem 2.2). Therefore if $\mathcal{J}_{[(\phi, z)]}$ denotes $\ker(\text{Ind}_{\mathbb{Z}_\phi}^{\mathbb{Z}}(\phi \times \gamma_z|_{\mathbb{Z}_\phi}) \circ q)$, then $\mathcal{F} = \{\mathcal{J}_{[(\phi, z)]} : \phi \in \Omega(A), z \in \mathbb{T}\}$, and the map $[(\phi, z)] \mapsto \mathcal{J}_{[(\phi, z)]}$ is homeomorphism of $\text{Prim}(A \times_\alpha \mathbb{Z}) \simeq \Omega(A) \times \mathbb{T} / \sim$ onto \mathcal{F} .

Proposition 3.7. *Let (A, α) be a system consisting of a separable abelian C^* -algebra A and an automorphism α of A . Then $A \times_\alpha^{\text{piso}} \mathbb{N}$ is GCR if and only if $\mathbb{Z} \backslash \Omega(A)$ is a T_0 space.*

Proof. By [9, Theorem 5.6.2], $A \times_\alpha^{\text{piso}} \mathbb{N}$ is GCR if and only if $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q$ and $A \times_\alpha \mathbb{Z} \simeq C_0(\Omega(A)) \times_\tau \mathbb{Z}$ are GCR. But since A is abelian, $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ is automatically CCR, and hence it is GCR. Therefore $A \times_\alpha^{\text{piso}} \mathbb{N}$ is GCR precisely when $A \times_\alpha \mathbb{Z}$ is GCR. By [12, Theorem 8.43], $A \times_\alpha \mathbb{Z}$ is GCR if and only if $\mathbb{Z} \backslash \Omega(A)$ is T_0 . \square

Proposition 3.8. *Let (A, α) be a system consisting of a separable abelian C^* -algebra A and an automorphism α of A . Then $A \times_\alpha^{\text{piso}} \mathbb{N}$ is not CCR.*

Proof. Note that $A \times_\alpha^{\text{piso}} \mathbb{N}$ is CCR if and only if $(B_{\mathbb{Z}} \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z} \simeq C_0(\Omega(B_{\mathbb{Z}} \otimes A)) \times_\tau \mathbb{Z}$ is CCR, because they are Morita equivalent (see [12, Proposition I.43]). Since for the \mathbb{Z} -orbit of a pair (m, ϕ) , we have

$$\overline{\mathbb{Z} \cdot (m, \phi)} = \overline{\mathbb{Z} \times \{\phi\}} = \overline{\mathbb{Z}} \times \overline{\{\phi\}} = (\mathbb{Z} \cup \{\infty\}) \times \{\phi\},$$

it follows that \mathbb{Z} -orbit of (m, ϕ) is not closed in $\Omega(B_{\mathbb{Z}} \otimes A) = (\mathbb{Z} \cup \{\infty\}) \times \Omega(A)$. Therefore by [12, Theorem 8.44], $C_0(\Omega(B_{\mathbb{Z}} \otimes A)) \times_\tau \mathbb{Z}$ is not CCR, and hence $A \times_\alpha^{\text{piso}} \mathbb{N}$ is not CCR. \square

Example 3.9. (Pimsner-Voiculescu Toeplitz algebra) Suppose $\mathcal{T}(A, \alpha)$ is the Pimsner-Voiculescu Toeplitz algebra associated to the system (A, α) (see [10]). It was shown in [4, §5] that $\mathcal{T}(A, \alpha)$ is isomorphic to the partial-isometric crossed product $A \times_{\alpha^{-1}}^{\text{piso}} \mathbb{N}$ associated to the system (A, α^{-1}) . Therefore when A is abelian and separable, the description of $\text{Prim}(\mathcal{T}(A, \alpha))$ follows completely from Theorem 3.5. In particular, for the trivial system (\mathbb{C}, id) , $\mathcal{T}(\mathbb{C}, \text{id})$ is the Toeplitz algebra $\mathcal{T}(\mathbb{Z})$ of integers isomorphic to $\mathbb{C} \times_{\text{id}}^{\text{piso}} \mathbb{N}$. So again by Theorem 3.5, $\text{Prim}(\mathcal{T}(\mathbb{Z}))$ corresponds to the disjoint union $\{0\} \sqcup \mathbb{T}$ in which every (nonempty) open set is of the form $\{0\} \cup W$ for some open subset W of \mathbb{T} . This description is known which coincides with the description of $\text{Prim}(\mathcal{T}(\mathbb{Z}))$ obtained from the well-known short exact sequence $0 \rightarrow \mathcal{K}(\ell^2(\mathbb{N})) \rightarrow \mathcal{T}(\mathbb{Z}) \rightarrow C(\mathbb{T}) \rightarrow 0$.

Example 3.10. Consider the system $(C(\mathbb{T}), \alpha)$ in which the action α is given by rotation through the angle $2\pi\theta$ with θ rational. By using the discussion in [12, Example 8.46], $\text{Prim}(C(\mathbb{T}) \times_\alpha^{\text{piso}} \mathbb{N})$ can be identified with the disjoint union

$$\mathbb{T} \sqcup \mathbb{T}^2,$$

in which by Theorem 3.5, each open set is given by

$$\begin{aligned} & \{U \subset \mathbb{T} : U \text{ is open in } \mathbb{T}\} \cup \\ & \{U \cup W : U \text{ is a nonempty open subset of } \mathbb{T}, \text{ and } W \text{ is open in } \mathbb{T}^2\}. \end{aligned}$$

Moreover the orbit space $\mathbb{Z} \backslash \mathbb{T}$ is homeomorphic to \mathbb{T} , which is obviously T_0 (in fact Hausdorff). So it follows by Proposition 3.7 that $C(\mathbb{T}) \times_{\alpha}^{\text{piso}} \mathbb{N}$ is GCR.

3.2. The topology of $\text{Prim}((B_{\mathbb{Z}} \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z})$ when A is separable and \mathbb{Z} acts on $\text{Prim } A$ freely. Consider a system (A, α) in which A is separable, and \mathbb{Z} acts on $\text{Prim } A$ freely. It follows that \mathbb{Z} acts on $\text{Prim}(B_{\mathbb{Z}} \otimes A)$ freely too. This is because, firstly, by [11, Theorem B.45], $\text{Prim}(B_{\mathbb{Z}} \otimes A)$ is homeomorphic to $\text{Prim } B_{\mathbb{Z}} \times \text{Prim } A$, and hence it is homeomorphic to $(\mathbb{Z} \cup \{\infty\}) \times \text{Prim } A$. Then \mathbb{Z} acts on $(\mathbb{Z} \cup \{\infty\}) \times \text{Prim } A$ such that

$$n \cdot (m, P) = (m + n, P) \text{ and } n \cdot (\infty, P) = (\infty, \alpha_n^{-1}(P))$$

for all $n, m \in \mathbb{Z}$ and $P \in \text{Prim } A$. Therefore the stability group of each (∞, P) equals the stability group \mathbb{Z}_P of P , which is $\{0\}$ as \mathbb{Z} acts on $\text{Prim } A$ freely, and stability group of each (m, P) is clearly $\{0\}$. So in the separable system $(B_{\mathbb{Z}} \otimes A, \mathbb{Z}, \beta \otimes \alpha^{-1})$ (with \mathbb{Z} abelian), \mathbb{Z} acts on $\text{Prim}(B_{\mathbb{Z}} \otimes A) \simeq (\mathbb{Z} \cup \{\infty\}) \times \text{Prim } A$ freely. Therefore by Theorem 2.4, $\text{Prim}((B_{\mathbb{Z}} \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z})$ is homeomorphic to the quasi-orbit space $\mathcal{O}(\text{Prim}(B_{\mathbb{Z}} \otimes A)) = \mathcal{O}((\mathbb{Z} \cup \{\infty\}) \times \text{Prim } A)$, which describes $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$ as well. We want to describe the quotient topology of $\mathcal{O}((\mathbb{Z} \cup \{\infty\}) \times \text{Prim } A)$ precisely, and identify the primitive ideals of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ coming from $\text{Prim}(A \times_{\alpha} \mathbb{Z})$. We have

$$\begin{aligned} \mathcal{O}(m, P) = \mathcal{O}(n, Q) &\iff \overline{\mathbb{Z} \cdot (m, P)} = \overline{\mathbb{Z} \cdot (n, Q)} \\ &\iff \overline{\mathbb{Z} \times \{P\}} = \overline{\mathbb{Z} \times \{Q\}} \\ &\iff \overline{\mathbb{Z} \times \{P\}} = \overline{\mathbb{Z} \times \{Q\}} \\ &\iff (\mathbb{Z} \cup \{\infty\}) \times \overline{\{P\}} = (\mathbb{Z} \cup \{\infty\}) \times \overline{\{Q\}}. \end{aligned}$$

Therefore $\mathcal{O}(m, P) = \mathcal{O}(n, Q)$ if and only if $\overline{\{P\}} = \overline{\{Q\}}$, and this happens precisely when $P = Q$ by the definition of the hull-kernel (Jacobson) topology on $\text{Prim } A$ (that is why the primitive ideal space of any C^* -algebra is always T_0 [9, Theorem 5.4.7]). So all pairs (m, P) for every $m \in \mathbb{Z}$ have the same quasi-orbit which can be parameterized by $P \in \text{Prim } A$, and since

$$\overline{\mathbb{Z} \cdot (\infty, Q)} = \overline{\{\infty\} \times \mathbb{Z} \cdot Q} = \overline{\{\infty\} \times \mathbb{Z} \cdot Q} = \{\infty\} \times \overline{\mathbb{Z} \cdot Q},$$

$\mathcal{O}(m, P) \neq \mathcal{O}(\infty, Q)$ for all $m \in \mathbb{Z}$ and $P, Q \in \text{Prim } A$. Moreover

$$\begin{aligned} \mathcal{O}(\infty, P) = \mathcal{O}(\infty, Q) &\iff \overline{\mathbb{Z} \cdot (\infty, P)} = \overline{\mathbb{Z} \cdot (\infty, Q)} \\ &\iff \{\infty\} \times \overline{\mathbb{Z} \cdot P} = \{\infty\} \times \overline{\mathbb{Z} \cdot Q}. \end{aligned}$$

Thus $\mathcal{O}(\infty, P) = \mathcal{O}(\infty, Q)$ if and only if $\overline{\mathbb{Z} \cdot P} = \overline{\mathbb{Z} \cdot Q}$, which means if and only if P and Q have the same quasi-orbit ($\mathcal{O}(P) = \mathcal{O}(Q)$) in $\mathcal{O}(\text{Prim } A) \simeq \text{Prim}(A \times_{\alpha} \mathbb{Z})$. So each quasi-orbit $\mathcal{O}(\infty, P)$ can be parameterized by the quasi-orbit $\mathcal{O}(P)$ in $\mathcal{O}(\text{Prim } A)$, and we can therefore identify $\mathcal{O}((\mathbb{Z} \cup \{\infty\}) \times \text{Prim } A)$ by the disjoint union

$$\text{Prim } A \sqcup \mathcal{O}(\text{Prim } A).$$

Then we have:

Theorem 3.11. *Let (A, α) be a system consisting of a separable C^* -algebra A and an automorphism α of A . Suppose that \mathbb{Z} acts on $\text{Prim } A$ freely. Then $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$ is*

homeomorphic to $\text{Prim } A \sqcup \mathcal{O}(\text{Prim } A)$, equipped with the (quotient) topology in which the open sets are of the form

$$\{U \subset \text{Prim } A : U \text{ is open in } \text{Prim } A\} \cup$$

$$\{U \cup W : U \text{ is a nonempty open subset of } \text{Prim } A, \text{ and } W \text{ is open in } \mathcal{O}(\text{Prim } A)\}.$$

Proof. Note that since by [12, Lemma 6.12], the quasi-orbit map $q : \text{Prim}(B_{\mathbb{Z}} \otimes A) \rightarrow \mathcal{O}(\text{Prim}(B_{\mathbb{Z}} \otimes A))$ is continuous and open, the proof follows from a similar argument to the proof of Theorem 3.5. So we skip it here. \square

Remark 3.12. Under the condition of Theorem 3.11, we want to identify the primitive ideals of $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$ coming from $\text{Prim}(A \times_{\alpha} \mathbb{Z})$, which form the closed subset

$$\mathcal{F} := \{\mathcal{J} \in \text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}) : \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q \subset \mathcal{J}\}$$

homeomorphic to $\text{Prim}(A \times_{\alpha} \mathbb{Z}) \simeq \mathcal{O}(\text{Prim } A)$ (see Theorem 2.4). These ideals are actually the kernels of the irreducible representations $(\text{Ind } \pi) \circ q = (\tilde{\pi} \times U) \circ q$ of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$, where π is an irreducible representation of A with $\ker \pi = P$. But since the pair $(\tilde{\pi}, U)$ is clearly a covariant partial-isometric representation of (A, α) , one can see that in fact, $(\text{Ind } \pi) \circ q = \tilde{\pi} \times^{\text{piso}} U$, where $\tilde{\pi} \times^{\text{piso}} U$ is the associated representation of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ correspondent to the pair $(\tilde{\pi}, U)$. Thus each element of \mathcal{F} is of the form $\ker(\tilde{\pi} \times^{\text{piso}} U)$ correspondent to the quasi-orbit $\mathcal{O}(P)$, and therefore we denote $\ker(\tilde{\pi} \times^{\text{piso}} U)$ by $\mathcal{J}_{\mathcal{O}(P)}$. So the map $\mathcal{O}(P) \rightarrow \mathcal{J}_{\mathcal{O}(P)}$ is a homeomorphism of $\mathcal{O}(\text{Prim } A)$ onto the closed subspace \mathcal{F} of $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$.

For the following remark, we need to recall that the primitive ideal space of any C^* -algebra A is locally compact [7, Corollary 3.3.8]. A locally compact space X (not necessarily Hausdorff) is called *almost Hausdorff* if each locally compact subspace U contains a relatively open nonempty Hausdorff subset (see [12, Definition 6.1.]). If a C^* -algebra is GCR, then it is almost Hausdorff (see the discussion on pages 171 and 172 of [12]). Finally if A is separable, then by applying [11, Theorem A.38] and [11, Proposition A.46], it follows that $\text{Prim } A$ is second countable.

Remark 3.13. It follows from [13] that if (A, \mathbb{Z}, α) is a separable system in which \mathbb{Z} acts on \hat{A} freely, then $A \times_{\alpha} \mathbb{Z}$ is GCR if and only if A is GCR and every \mathbb{Z} -orbit in \hat{A} is discrete. But every \mathbb{Z} -orbit in \hat{A} is discrete if and only if for each $[\pi] \in \hat{A}$, the map $\mathbb{Z} \rightarrow \mathbb{Z} \cdot [\pi]$ defined by $n \mapsto n \cdot [\pi] = [\pi \circ \alpha_n^{-1}]$ is a homeomorphism, and this statement itself, by [12, Theorem 6.2 (Mackey-Glimm Dichotomy)], is equivalent to saying that the orbit space $\mathbb{Z} \backslash \hat{A}$ is T_0 . Therefore we can rephrase the statement of [13] to say that if (A, \mathbb{Z}, α) is a separable system in which \mathbb{Z} acts on \hat{A} freely, then $A \times_{\alpha} \mathbb{Z}$ is GCR if and only if A is GCR and the orbit space $\mathbb{Z} \backslash \hat{A}$ is T_0 .

Proposition 3.14. *Let (A, α) be a system consisting of a separable C^* -algebra A and an automorphism α of A . Suppose that \mathbb{Z} acts on \hat{A} freely. Then $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is GCR if and only if A is GCR and the orbit space $\mathbb{Z} \backslash \hat{A}$ is T_0 .*

Proof. The proof follows from a similar argument to the proof of Proposition 3.7 and Remark 3.13. \square

Example 3.15. Consider the system $(C(\mathbb{T}), \alpha)$ in which the action α is given by rotation through the angle $2\pi\theta$ with θ irrational. Then \mathbb{Z} acts on $\text{Prim}(C(\mathbb{T})) = C(\mathbb{T})^{\wedge} = \mathbb{T}$

freely (see [12, Example 8.45] or [6, Example 10.18]). Therefore by Theorem 3.11, $\text{Prim}(C(\mathbb{T}) \times_{\alpha}^{\text{piso}} \mathbb{N})$ can be identified with the disjoint union $\mathbb{T} \sqcup \mathcal{O}(\mathbb{T})$. But the quasi-orbit space $\mathcal{O}(\mathbb{T})$ contains only one point as each \mathbb{Z} -orbit is dense in \mathbb{T} (see [12, Lemma 3.29]). Let us parameterize this only point by 0 (note that $\mathcal{O}(\mathbb{T})$ is homeomorphic to the primitive ideal space of the irrational rotation algebra $A_{\theta} := C(\mathbb{T}) \times_{\alpha} \mathbb{Z}$ which is simple). So $\text{Prim}(C(\mathbb{T}) \times_{\alpha}^{\text{piso}} \mathbb{N})$ is actually identified with

$$\mathbb{T} \sqcup \{0\},$$

where each open set is given by

$$\{U \subset \mathbb{T} : U \text{ is open in } \mathbb{T}\} \cup \{U \cup \{0\} : U \text{ is a nonempty open subset of } \mathbb{T}\}.$$

Here we would like to mention that 0 in $\mathbb{T} \sqcup \{0\}$ corresponds to the primitive ideal $\mathcal{K}(\ell^2(\mathbb{N})) \otimes C(\mathbb{T})$ of $C(\mathbb{T}) \times_{\alpha}^{\text{piso}} \mathbb{N}$. Finally, although $C(\mathbb{T})$ is GCR (in fact CCR), the orbit space $\mathbb{Z} \backslash \mathbb{T}$ is not T_0 as each \mathbb{Z} -orbit is dense in \mathbb{T} . So it follows by Proposition 3.14 that $C(\mathbb{T}) \times_{\alpha}^{\text{piso}} \mathbb{N}$ is not GCR.

Remark 3.16. Recall that since the Pimsner-Voiculescu Toeplitz algebra $\mathcal{T}(A, \alpha)$ is isomorphic to $A \times_{\alpha^{-1}}^{\text{piso}} \mathbb{N}$ (see Example 3.9), if A is separable and \mathbb{Z} acts on $\text{Prim } A$ freely, then the description of $\text{Prim}(\mathcal{T}(A, \alpha))$ is obtained completely from Theorem 3.11.

4. PRIMITIVITY AND SIMPLICITY OF $A \times_{\alpha}^{\text{piso}} \mathbb{N}$

In this section, we want to discuss the primitivity and simplicity of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$. Recall that a C^* -algebra is called *primitive* if it has a faithful nonzero irreducible representation, and it is called *simple* if it has no nontrivial ideal.

Theorem 4.1. *Let (A, α) be a system consisting of a C^* -algebra A and an automorphism α of A . Then $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is primitive if and only if A is primitive.*

Proof. If $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is primitive, it has a faithful nonzero irreducible representation $\rho : A \times_{\alpha}^{\text{piso}} \mathbb{N} \rightarrow B(\mathcal{H})$. Then since the restriction of ρ to the ideal $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker \rho$ is nonzero, it gives an irreducible representation of $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ which is clearly faithful. So it follows that $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ is primitive, and therefore A must be primitive as well.

Conversely, if A is primitive, then it has a faithful nonzero irreducible representation π on some Hilbert space H ($P = \ker \pi = \{0\}$). We show that the associated irreducible representation $(\Pi \times V)_P$ of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ on $\ell^2(\mathbb{N}, H)$ is faithful. By [8, Theorem 4.8], it is enough to see that if $\Pi(a)(1 - V^*V) = 0$, then $a = 0$. If $\Pi(a)(1 - V^*V) = 0$, then

$$\Pi(a)(1 - V^*V)(e_0 \otimes h) = (e_0 \otimes \pi(a)h) = 0 \quad \text{for all } h \in H.$$

It follows that $\pi(a)h = 0$ for all $h \in H$, and therefore $\pi(a) = 0$. Since π is faithful, we must have $a = 0$. This completes the proof. \square

Remark 4.2. Note that Theorem 4.1 simply means that in the homeomorphism $P \mapsto \mathcal{I}_P$ mentioned in Remark 3.2, P is the zero ideal if and only if \mathcal{I}_P is the zero ideal. This is because if $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is primitive, then its zero ideal as one of its primitive ideals is of the form \mathcal{I}_P (coming from $\text{Prim } A$), as $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A \neq 0$.

Finally it is not difficult to see that $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is not simple. This is because as we see, it contains $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ as a nonzero ideal. Moreover if $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A = A \times_{\alpha}^{\text{piso}} \mathbb{N}$, then $A \times_{\alpha} \mathbb{Z} \simeq (A \times_{\alpha}^{\text{piso}} \mathbb{N}) / (\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$ must be the zero algebra. So it follows that $A = 0$, which is a contradiction as we have $A \neq 0$. Therefore $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ contains $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$ as a proper nonzero ideal, and hence we have proved the following:

Theorem 4.3. *Let (A, α) be a system consisting of a C^* -algebra A and an automorphism α of A . Then $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ is not simple.*

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